Games of Elimination

Dmitry Ilinsky,
Sergei Izmalkov,
Alexei Savvateev

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HSE
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Agents are modelled as “anonymous” (*):

- infinitely small in competitive markets;
- own actions (e.g. price or quantity) in duopolies;
- own action (e.g. enter/quit) in the war of attrition, IO games.
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Our target: model interactions with actions that may be directed at a specific opponent.
Examples: political games, mafia control, negative ads, litigation, patent races, industrial espionage, ...
Games of Elimination

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- Multi-round tournaments with actions that may eliminate competitors.
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- We analyze games with 2 and 3+ players.
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We analyze games with 2 and 3+ players.

Cooperation (!) may arise even in these games — in the face of death.
The model

- $N$ players, $i$ with marksmanship $\alpha_i \in (0, 1)$. 
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- In each round:
  each “alive” player selects a target: another player or “the air”
  all shoot simultaneously
  those that are shot are eliminated

Payoffs: if $K$ players are “alive” at the end, each receives $X^K$, others 0.

$Y = X_1 > X_2 > \cdots > X_N$. For $i > 1$, $X_i < Y/i$. 

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  OR when all shot in the air.
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The duel

Player $i$’s payoff

$$Z_i = (1 - \alpha_j)\alpha_i Y + (1 - \alpha_j)(1 - \alpha_i)Z_i;$$

$$Z_i = \frac{\alpha_i - \alpha_i\alpha_j}{\alpha_i + \alpha_j - \alpha_i\alpha_j} Y. \quad (1)$$

In the case $\alpha_i = \alpha_j = \alpha$ the payoff is $Z = \frac{1 - \alpha}{2 - \alpha} Y$.

Note that $Z_i$ is the payoff that player $i$ can guarantee to herself no matter what is the strategy of the opponent.
Lemma

“Peace,” that is simultaneous shooting in the air, cannot be sustained in equilibrium.

\[
D_i + D_j = \frac{\alpha_i + \alpha_j}{\alpha_i + \alpha_j - \alpha_i \alpha_j} \quad Y > Y > X_2 + X_2.
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$$D_i + D_j = \frac{\alpha_i + \alpha_j}{\alpha_i + \alpha_j - \alpha_i \alpha_j} Y > Y > X_2 + X_2.$$

- Note, that if $X_2 > Y/2$, peace can be sustained, but not with “strong” players.
There exist two asymmetric SPE that (weakly) Pareto dominate the “war” equilibrium.
Cooperation

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- Players shoot in sequence, player $i$ in odd periods, player $j$ in even ones.
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\[
A_{i1} = \alpha_i Y + (1 - \alpha_j)(1 - \alpha_i)A_i, \quad \text{(2)}
\]
\[
A_{j2} = (1 - \alpha_i)\alpha_j Y + (1 - \alpha_j)(1 - \alpha_i)A_j. \quad \text{(3)}
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Note that for all $i$, $A_{i2} = Z_i$, and so $A_{i1} = Y - Z_j$. 
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- By shooting in sequence, the players eliminate the undesirable event of both of them dying.
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▶ By shooting in sequence, the players eliminate the undesirable event of both of them dying.

▶ This may explain why some duels have “sequential” rules.
Lemma

The set of pure SPE is completely described by the pair \((T, k)\), where \(T \in \mathbb{N} \cup \{\infty\}\) is the period in which “polite war” starts, and \(k\) is the player whose turn is to shoot first. Before period \(T\) the players follow the strategies of the “war” equilibrium.
Other equilibria

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- Mixed equilibria: ONE player mixes and can trigger the “polite war.”
Truels

3 players, $\alpha \geq \beta \geq \gamma$. 
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- **Thm**: The one-shot deviation principle.
Stationary equilibria

- Stationary equilibria — “alive” players’ strategies depend only on the set of “alive” players.
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- The following pure stationary SPE may exist (under certain conditions on $\alpha, \beta, \gamma$).

\[
\begin{align*}
\alpha & \quad \beta \\
\gamma & \quad \beta \\
\alpha & \quad \gamma \\
\alpha & \quad \gamma
\end{align*}
\]
Non-stationary equilibria

“Efficient war”

Various types of “polite war” also exist.
Peace sustainable?

- $X_2 = X_3 = 0$: pure truel.
  Peace cannot be sustained.
  Never optimal to “abstain” in stationary equilibria.
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- $X_2 = X_3 = 0$: pure truel.
  Peace cannot be sustained.
  Never optimal to “abstain” in stationary equilibria.
- $X_3 > 0$.
  Peace can be sustained.
  Easier for strong players (!).
  Non-monotone condition (!).
Cooperation

Three kinds of cooperation may emerge

1. Peace: payoffs to peace have to be sufficiently high. Harder to sustain with strong players when $N = 2$, easier when $N > 2$. 
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2. “Polite” and “efficient war”: players eliminate a “no one survives” event.
Cooperation

Three kinds of cooperation may emerge

1. Peace: payoffs to peace have to be sufficiently high. Harder to sustain with strong players when \( N = 2 \), easier when \( N > 2 \).

2. “Polite” and “efficient war”: players eliminate a “no one survives” event.

3. Cycles: players select different targets to avoid duplication of effort.
Conclusions and Future directions

- Interesting equilibrium patterns.
- Various kinds of cooperation emerge.
- For $N = 3$, peace is hardest to achieve for “intermediate” shooters.
- “Weak” players may have largest payoffs, and benefit from their opponents getting stronger.
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- Costly actions/ abilities.
- Experiments.